Journal of Statistical Physics, Vol. 116, Nos. 1/4, August 2004 (© 2004)

# On the Relation between Strong Subadditivity and Entanglement

**Heide Narnhofer**[1] **and Walter Thirring**[1]

*Received June 3, 2003; accepted July 21, 2003*

Strong subadditivity is used to improve the triangular inequality for the entropy of tensorproducts by the amount of entanglement.

**KEY WORDS:** Entanglement.

## 1. INTRODUCTION

Classically the entropy of a discrete system exceeds the entropy of any of is parts but in quantum theory this need not be the case. If a system is split into a part $\mathscr{A}$ and a rest $\mathscr{B}$ such that the observables belong to $\mathscr{A} \otimes \mathscr{B}$ then generally we have for the entropy of the total system the following bounds:[1, 2, 5]

Classically:

$$\max\{S(\mathscr{A}), S(\mathscr{B})\} \leqslant S(\mathscr{A} \otimes \mathscr{B}) \leqslant S(\mathscr{A}) + S(\mathscr{B}) \tag{1}$$

quantum:

$$|S(\mathscr{A}) - S(\mathscr{B})| \leqslant S(\mathscr{A} \otimes \mathscr{B}) \leqslant S(\mathscr{A}) + S(\mathscr{B}). \tag{2}$$

Thus in quantum theory a state $\omega$ over $\mathscr{A} \otimes \mathscr{B}$ may contain correlations which are masked in $\omega|_{\mathscr{A}}$ and $\omega|_{\mathscr{B}}$. In this paper we want to show that the triangular inequality does not give justice to these quantum correlations. We will demonstrate that it can be improved by using the so called entanglement of formation $E$. The mathematical expression appeared previously

[1] Institut für Theoretische Physik, Universität Wien; e-mail: narnh@ap.univie.ac.at



0022-4715/04/0800-1189/0 

in other contexts,[6, 7] though the relation to entanglement was first given in ref. 8. This improvement reeds

$$|S(\mathscr{A})-S(\mathscr{B})| \leqslant \max\{S(\mathscr{A}), S(\mathscr{B})\}-E \leqslant S(\mathscr{A}\otimes\mathscr{B}). \tag{3}$$

Further we will show that this is always an improvement unless the triangular inequality becomes an equality. This shows that $E$ gives the maximal violation of the monotonicity of the entropy. There are other measurements for entanglement, e.g., the entanglement of distillation,[8] the Hilbert Schmidt distance of a state from the set of separable states,[9] for other possibilities see the reviews in refs. 10 and 11. But the entanglement of formation is best related to the entropy.

The entanglement inequality turns out to be equivalent to the strong subadditivity with respect to a suitable third party, the so called abelian model.[7] Recently states for which the strong subadditivity becomes sharp have been characterized.[12] In this situation also the entanglement inequality becomes sharp. Thus we can give examples in which the entanglement inequality is sharp whereas the triangular inequality is not.

It is for us a pleasure to dedicate this note to E. H. Lieb, who gave together with M. B. Ruskai[3, 4] the first proof of strong subadditivity.

## 2. THE ESTIMATES

We consider the tensor product of two finite dimensional full matrix algebras $\mathscr{M}=\mathscr{A}\otimes\mathscr{B}$. The entanglement of formation of a state $\omega$ over $\mathscr{M}$ with respect to the subalgebra $\mathscr{A}$ is defined as

$$E(\mathscr{M},\mathscr{A},\omega)=\inf_{\omega=\sum_i^N \lambda_i\omega_i}\sum_i^N \lambda_i S(\omega_i,\mathscr{A})=E(\mathscr{M},\mathscr{B},\omega) \tag{4}$$

where the decomposition is considered as decomposition into states over $\mathscr{M}$ and $S(\omega_i,\mathscr{A})$ always means the entropy of the state $\omega_i$ considered as state over $\mathscr{A}$. By concavity of the entropy the infimum is reached for a decomposition into pure states over $\mathscr{M}$. For these states $S(\omega_i,\mathscr{A})=S(\omega_i,\mathscr{B})$ and guarantees therefore the above equality. In ref. 7 the notion of an abelian model was introduced: if we have a decomposition $\omega=\sum_i^N\lambda_i\omega_i$ into $N$ states then we can take an abelian algebra $\mathscr{C}=(P_1,..., P_N)$ and can characterize the decomposition of the state $\omega$ by introducing a state $\Lambda$ over $\mathscr{M}\otimes\mathscr{C}$ with $\Lambda(M\otimes P_i)=\lambda_i\omega_i(M)$ so that $\Lambda|_{\mathscr{M}}=\omega$. Since with this characterization there is a one-to-one correspondence between decompositions and abelian models we can express the entanglement of a subalgebra $\mathscr{A} \subset \mathscr{M}$ as infimum over abelian models

$$E(\mathscr{M}, \mathscr{A}, \omega) = \inf_{\mathscr{C}, \Lambda} (S(\Lambda, \mathscr{A} \otimes \mathscr{C}) - S(\Lambda, \mathscr{C})). \tag{5}$$

Notice that this expression is $\geqslant 0$, because the algebra $\mathscr{C}$ is abelian. It is well defined for every subalgebra, but here we will mainly consider $\mathscr{M} = \mathscr{A} \otimes \mathscr{B}$. Since we have to decompose into pure states over $\mathscr{M}$ to reach the infimum (for finite dimensional matrices the infimum is in fact a minimum by standard arguements on the simplex structure of state space) the state $\Lambda$ corresponds to a density matrix over $\mathscr{H}_{\mathscr{M}} \otimes \mathscr{H}_{\mathscr{C}}$, namely to $\rho(\Lambda) = \sum_i^N \lambda_i Q_i \otimes P_i$ where $P_i$, $Q_i$ are a set of one dimensional projections in $\mathscr{C}$ resp. in $\mathscr{M}$. The $Q_i$ are in general not orthogonal, but the $P_i$ and therefore also the $Q_i \otimes P_i$ are. This implies that $S(\Lambda, \mathscr{M} \otimes \mathscr{C}) = S(\Lambda, \mathscr{C})$. Furthermore $\Lambda|_{\mathscr{M}} = \omega$ and $S(\Lambda, \mathscr{M} \otimes \mathscr{C}) \geqslant S(\Lambda, \mathscr{M}) = S(\omega, \mathscr{M})$, if $\Lambda$ is a decomposition into pure states over $\mathscr{M}$. If $\Lambda$ is the optimal decomposition for $E(\mathscr{M}, \mathscr{A}, \omega)$ we even have $S(\Lambda, \mathscr{A} \otimes \mathscr{C}) \geqslant S(\Lambda, \mathscr{M} \otimes \mathscr{C})$, in flagrant violation of monotonicity. Strong subadditivity tells us that

$$S(\Lambda, \mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C}) + S(\Lambda, \mathscr{C}) \leqslant S(\Lambda, \mathscr{A} \otimes \mathscr{C}) + S(\Lambda, \mathscr{B} \otimes \mathscr{C}). \tag{6}$$

The separable states are those states that can be written as convex combination of tensor products, therefore those states for which $E(\mathscr{M}, \mathscr{A}, \omega) = 0$. In this situation the optimal decomposition is the one into states that are also pure on $\mathscr{A}$ and $\mathscr{B}$. Therefore we obtain

**Lemma 1.** If the state $\Lambda$ describes an optimal decomposition with respect to the entanglement of $\mathscr{A}$ then the inequality (6) becomes an equality if and only if the state is separable.

*Proof.*

$$\begin{aligned} &S(\Lambda, \mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C}) + S(\Lambda, \mathscr{C}) \\ &\quad = 2S(\Lambda, \mathscr{C}) \leqslant S(\Lambda, \mathscr{A} \otimes \mathscr{C}) + S(\Lambda, \mathscr{B} \otimes \mathscr{C}) \\ &\quad = E(\mathscr{A}) + S(\Lambda, \mathscr{C}) + E(\mathscr{B}) + S(\Lambda, \mathscr{C}). \end{aligned} \tag{7}$$

Thus equality holds iff $E(\mathscr{A}) = E(\mathscr{B}) = 0$.

Strong subadditivity also offers another inequality, namely

$$S(\Lambda, \mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C}) + S(\Lambda, \mathscr{B}) \leqslant S(\Lambda, \mathscr{A} \otimes \mathscr{B}) + S(\Lambda, \mathscr{B} \otimes \mathscr{C}). \tag{8}$$

Therefore

**Lemma 2.** If the state is separable, then

$$S(\omega, \mathscr{A} \otimes \mathscr{B}) \geqslant \max(S(\omega, \mathscr{B}), S(\omega, \mathscr{A})). \tag{9}$$

**Remark.** Separable states are those states that are only classically correlated, i.e., they can be written as convex combination of tensor products $\omega = \sum_i \phi_{i\mathscr{A}} \otimes \phi_{i\mathscr{B}}$. We see that they inherit the monotonicity of the entropy in the classical theory.

*Proof.* (9) can be proven in several ways. Here we want to demonstrate that it is also a consequence of strong subadditivity applied to an appropriate choice of algebras. The combination of strong subadditivity with respect to $\mathscr{B}$ and additivity with respect to $\mathscr{C}$ gives the desired result. More explicitly, by (6) taken as equality according to Lemma (1) for the optimal $\Lambda$, inserting it in the subadditivity inequality, we obtain

$$\begin{aligned} &S(\Lambda, \mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C}) + S(\Lambda, \mathscr{B}) \\ &\quad = S(\Lambda, \mathscr{A} \otimes \mathscr{C}) - S(\Lambda, \mathscr{C}) + S(\Lambda, \mathscr{B} \otimes \mathscr{C}) + S(\Lambda, \mathscr{B}) \\ &\quad \leqslant S(\Lambda, \mathscr{A} \otimes \mathscr{B}) + S(\Lambda, \mathscr{B} \otimes \mathscr{C}) \end{aligned} \tag{10}$$

and thus reduces to the inequality (9) if we use monotonicity of the entropy for an abelian algebra in the tensor product, i.e., $S(\Lambda, \mathscr{A} \otimes \mathscr{C}) - S(\Lambda, \mathscr{C}) \geqslant 0$. The same holds for $\mathscr{A} \leftrightarrow \mathscr{B}$.

**Remark.** The converse is false. $S$ may be monotonic even if $\omega$ is entangled. An example is given by the Werner states for $\mathscr{A} = \mathscr{B} = M_2$. With $\rho_\alpha = \frac{1}{4}(1 - \alpha \vec{\sigma}_{\mathscr{A}} \otimes \vec{\sigma}_{\mathscr{B}})$ the states are separable for $-\frac{1}{3} \leqslant \alpha \leqslant \frac{1}{3}$. $S(\rho_\alpha, \mathscr{A}) = S(\rho_\alpha, \mathscr{B}) = \ln 2$ whereas

$$S(\rho_\alpha) = -\frac{1}{4}(1+3\alpha) \ln \frac{1+3\alpha}{4} - \frac{3}{4}(1-\alpha) \ln \frac{1-\alpha}{4} = \ln 2 + \frac{1}{2} \ln 3 + \mathcal{O}(\epsilon) > \ln 2$$

if $\alpha = \frac{1}{3} + \epsilon$ and $\epsilon$ is sufficiently small. For $\alpha = 1$ the state becomes pure, $S(\rho_1) = 0$ and the inequality fails.

The optimal decomposition for which also $S(\Lambda, \mathscr{M}) = S(\Lambda, \mathscr{C})$ can be used to sharpen the lower bound on the entropy of $\mathscr{A} \otimes \mathscr{B}$ also in the general situation:

**Lemma 3 (Entanglement Inequality).** The entropy satisfies

$$S(\omega, \mathscr{A} \otimes \mathscr{B}) \geqslant H(\mathscr{B}, \omega) \equiv S(\omega, \mathscr{B}) - E(\mathscr{A} \otimes \mathscr{B}, \mathscr{B}, \omega) \tag{11}$$

and the same with $\mathscr{A} \leftrightarrow \mathscr{B}$.

*Proof.* From the strong subadditivity we conclude for an optimal decomposition $\Lambda$

$$\begin{aligned} S(\Lambda, \mathscr{A} \otimes \mathscr{B}) = S(\omega, \mathscr{A} \otimes \mathscr{B}) &\geqslant S(\Lambda, \mathscr{B}) + S(\Lambda, \mathscr{C}) - S(\Lambda, \mathscr{B} \otimes \mathscr{C}) \\ &= S(\omega, \mathscr{B}) - E(\mathscr{A} \otimes \mathscr{B}, \mathscr{B}, \omega) \equiv H(\mathscr{B}, \omega). \end{aligned} \tag{12}$$

**Lemma 4.** Let $\omega$ be a state over $\mathscr{M} = \mathscr{A} \otimes \mathscr{B}$. Then the following properties are equivalent:

(i) $\omega$ is pure or $S(\omega, \mathscr{A}) = 0$.

(ii) The triangular inequality $S(\omega, \mathscr{A} \otimes \mathscr{B}) \geqslant S(\omega, \mathscr{B}) - S(\omega, \mathscr{A})$ becomes an equality.

(iii) The entanglement inequality

$$S(\omega, \mathscr{A} \otimes \mathscr{B}) \geqslant H(\mathscr{A}, \omega) \equiv S(\omega, \mathscr{A}) - E(\mathscr{A} \otimes \mathscr{B}, \mathscr{A}, \omega) \tag{13}$$

is not a strict sharpening of the triangular inequality. (For any $\omega$ it is at least as good as the triangular inequality, because $E \leqslant S(\mathscr{B})$.)

**Remark.** We formulate the inequalities in one ordering. Of course in the inequalities the algebras $\mathscr{A}$ and $\mathscr{B}$ can be exchanged. But it is possible that the inequalities are sharp with respect to one ordering but not with respect to the other.

*Proof.*

(i) $\rightarrow$ (ii): For $\omega$ pure this is a well known result of entropy theory. Similarly $\omega$ pure on $\mathscr{A}$ implies $S(\omega, \mathscr{A} \otimes \mathscr{B}) = S(\omega, \mathscr{B})$.

(ii) $\rightarrow$ (i) : We consider a state $\omega$ where $S(\omega, \mathscr{A} \otimes \mathscr{B}) = S(\omega, \mathscr{B}) - S(\omega \mathscr{A})$. But in general $S(\omega, \mathscr{A} \otimes \mathscr{B}) \geqslant S(\omega, \mathscr{B}) - E(\mathscr{A} \otimes \mathscr{B}, \mathscr{B}, \omega)$ and also $S(\omega, \mathscr{A}) \geqslant E(\mathscr{A} \otimes \mathscr{B}, \mathscr{B}, \omega)$ it follows that $S(\omega, \mathscr{A}) = E(\mathscr{A}, \omega)$ This holds only if the decomposition of $\omega$ is not felt by $\mathscr{A}$. This either happens when $\omega$ is pure or when $S(\omega, \mathscr{A}) = 0$, i.e., in (i).

(i) $\leftrightarrow$ (iii): The entanglement inequality does not refine the triangular inequality iff $E(\mathscr{A}, \omega) = S(\omega, \mathscr{A})$.

Finally we consider, whether the class of states for which the entanglement inequality (11) becomes an equality is larger than the class of states for which the triangular inequality is an equality. The former happens when the strong subadditivity with respect to $\mathscr{B}$ becomes an equality. Here we use the following result of ref. 12

**Theorem.** We consider the tensor product of three algebras, $\mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C} \equiv \mathscr{A}\mathscr{B}\mathscr{C}$, not necessarily matrix algebras or abelian algebras. For a state $\omega_{\mathscr{A}\mathscr{B}\mathscr{C}}$ corresponding to a density matrix $\rho_{\mathscr{A}\mathscr{B}\mathscr{C}}$ strong subadditivity becomes additivity

$$S(\omega_{\mathscr{A}\mathscr{B}\mathscr{C}})+S(\omega_{\mathscr{B}})=S(\omega_{\mathscr{A}\mathscr{B}})+S(\omega_{\mathscr{B}\mathscr{C}}) \tag{14}$$

iff the Hilbert space $\mathscr{H}_{\mathscr{B}}$ on which $\mathscr{B}$ acts can be decomposed into a direct tensor product

$$\mathscr{H}_{\mathscr{B}}=\bigoplus_j \mathscr{H}_{b_j^L} \otimes \mathscr{H}_{b_j^R} \tag{15}$$

such that

$$\rho_{\mathscr{A}\mathscr{B}\mathscr{C}}=\bigoplus_j q_j \rho_{\mathscr{A}b_j^L} \otimes \rho_{b_j^R\mathscr{C}} \tag{16}$$

where $\rho_{\mathscr{A}b_j^L}$ is a density matrix over the Hilbert space $\mathscr{H}_{\mathscr{A}} \otimes \mathscr{H}_{b_j^L}$ and $\rho_{b_j^R\mathscr{C}}$ is a density matrix over the Hilbert space $\mathscr{H}_{b_j^R} \otimes \mathscr{H}_{\mathscr{C}}$ and $\{q_j\}$ is a probability distribution.

If we translate this condition to our special situation where $\mathscr{C}$ is abelian and corresponds to a decomposition into pure states over $\mathscr{A} \otimes \mathscr{B}$ then we write $\rho_{b_j^R\mathscr{C}}=\sum_i^N \mu_{ij}\rho_{b_j^R} \otimes P_i$. Taking the expectation value with $P_i$ it follows that

$$\bigoplus_j q_j \mu_{ij} \rho_{\mathscr{A}b_j^L} \otimes \rho_{b_j^R} \tag{17}$$

is a one dimensional projector. Therefore the sum must contain only one term and $\mu_{ij}$ must reduce to a Kronecker $\delta$ and also the individual $\rho_{\mathscr{A}b_j^L}$ and $\rho_{b_j^R}$ must be one dimensional projectors. Therefore we can apply the above theorem in

**Lemma 5.** The entanglement inequality becomes an equality for $H(\mathscr{B})$ if the density matrix over $\mathscr{A} \otimes \mathscr{B}$ can be written as

$$\rho_{\mathscr{A}\otimes\mathscr{B}}=\sum_j q_j \left|\sum_i \mu_{ij}\phi_{ij} \otimes \psi_{ij}\right\rangle\left\langle \sum_i \mu_{ij}\phi_{ij} \otimes \psi_{ij}\right| \tag{18}$$

where $|\phi_{ij}\rangle$ are normalized vectors in $\mathscr{H}_{\mathscr{A}}$ and $|\psi_{ij}\rangle$ $\forall(ij)$ are orthonormal vectors in $\mathscr{H}_{\mathscr{B}}$.

In the situation of equality for the triangular inequality (Lemma 4(ii)) the condition reduces to the fact that either there is no summation over $j$ and in addition also the $\phi_{ij}$ are orthonormal or that there is no summation over $i$ and all the $\phi_j$ are identical. Both are restrictions of the more general situation in the lemma.

In the special example of $\mathscr{A}=M_2$, we have the following result: The entanglement inequality (11) reduces to an equality only if $\omega$ is pure on $\mathscr{A}$ or pure on $\mathscr{A}\otimes\mathscr{B}$ or if $\omega$ is separable, i.e., if $\rho=\lambda P_1\otimes|\uparrow\rangle\langle\uparrow|+(1-\lambda)\,P_2\otimes|\downarrow\rangle\langle\downarrow|$ with $P_1$, $P_2$ arbitrary one dimensional projectors in $\mathscr{A}$. Remember, that the algebras $\mathscr{A}$ and $\mathscr{B}$ can behave differently, when we have formulated the entanglement inequality with respect to one.

If the entanglement inequality becomes an equality, we can make another observation: in the non commutative case a replacement for the monotonicity of the entropy is furnished by the conditional entropy.

**Definition.** Given a state $\omega$ over an algebra $\mathscr{M}$. Let $\mathscr{A}$ and $\mathscr{B}$ be subalgebras of $\mathscr{M}$. Then the conditional entropy of $\mathscr{B}$ with respect to $\mathscr{A}$ is defined via the relative entropy $S(\rho\,|\,\sigma)=\operatorname{tr}\sigma(\ln\sigma-\ln\rho)$ by

$$H(\mathscr{B}\,|\,\mathscr{A})_\omega=\sup\sum_j \mu_j(S(\omega\,|\,\omega_j)|_{\mathscr{B}}-S(\omega\,|\,\omega_j)|_{\mathscr{A}})\geqslant 0$$

where the supremum is taken over all decompositions of $\omega$ into states over $\mathscr{M}$, $\omega=\sum_j\mu_j\omega_j$, and the lower bound ($\geqslant 0$)is obtained if the state is not decomposed.

**Remark.** There is another definition for the conditional entropy, namely $S(\rho_{\mathscr{A}\otimes\mathscr{B}})-S(\rho_{\mathscr{A}})$. However, the conditional entropy in our definition is positive and can be understood as the gain of information that we get from $\mathscr{A}$ if we already know $\mathscr{B}$. It is defined in a larger context than for tensor products, not trivially related to other expression and widely used in the mathematical literature.[5] If all algebras are abelian (which we indicate by the index 0) then for the tensor product $H(\mathscr{B}_0\,|\,\mathscr{A}_0)=H(\mathscr{A}_0\otimes\mathscr{B}_0\,|\,\mathscr{A}_0)=S(\mathscr{A}_0\otimes\mathscr{B}_0)-S(\mathscr{A}_0)$. In the non abelian situation we have the following inequality as alternative to the subadditivity (it can be better, e.g., if $\omega$ is pure on $\mathscr{A}\otimes\mathscr{B}$, but also worse, e.g., if $\mathscr{A}=\mathscr{B}\otimes\mathscr{B}$ and $\omega$ is the tracial state):

**Lemma 6.**

$$S(\mathscr{A} \otimes \mathscr{B}) \leqslant H(\mathscr{A}) + H(\mathscr{A} \otimes \mathscr{B} \mid \mathscr{A}). \tag{19}$$

This follows from

$$\sup \sum_j \mu_j (S(\omega \mid \omega_j)|_{\mathscr{A} \otimes \mathscr{B}} - S(\omega \mid \omega_j)|_{\mathscr{A}}) + \sup \sum_j \mu_j S(\omega \mid \omega_j)|_{\mathscr{A}}$$

$$\geqslant \sup \sum_j \mu_j (S(\omega \mid \omega_j)|_{\mathscr{A} \otimes \mathscr{B}}) = S(\omega, \mathscr{A} \otimes \mathscr{B}). \tag{20}$$

Of course it would be desirable to see some relation with $H(\mathscr{B} \mid \mathscr{A})$ instead of $H(\mathscr{A} \otimes \mathscr{B} \mid \mathscr{A})$. But it is difficult to control the expression since neither monotonicity between the algebras nor convexity properties can be applied.

In the special situation of

$$\rho_{\mathscr{A} \otimes \mathscr{B}} = \sum_j q_j \left| \sum_i \mu_{ij} \phi_{ij} \otimes \psi_{ij} \right\rangle \left\langle \sum_i \mu_{ij} \phi_{ij} \otimes \psi_{ij} \right| \tag{21}$$

where strong subadditivity becomes equality we observe

$$S(\omega \mid \omega_j)|_{\mathscr{B}} = S(\omega \mid \omega_j)|_{\mathscr{A} \otimes \mathscr{B}} \tag{22}$$

for any $\omega_j$ contributing to a decomposition. Therefore this equality together with monotonicity $\mathscr{A} \subset \mathscr{A} \otimes \mathscr{B}$ implies that for such $\omega$

$$H(\mathscr{A} \mid \mathscr{B}) = H(\mathscr{A} \mid \mathscr{A} \otimes \mathscr{B}) = 0 \tag{23}$$

$$H(\mathscr{B} \mid \mathscr{A}) = H(\mathscr{A} \otimes \mathscr{B} \mid \mathscr{A}) = S(\mathscr{A} \otimes \mathscr{B}) - H(\mathscr{A}). \tag{24}$$

Though the entanglement inequality becomes an equality only with respect to one subalgebra we have the equality

$$S(\mathscr{A} \otimes \mathscr{B}) = H(\mathscr{A}) + H(\mathscr{B} \mid \mathscr{A}) = H(\mathscr{B}) + H(\mathscr{A} \mid \mathscr{B}). \tag{25}$$

This covers the case where $\mathscr{A}$ is abelian, but again we have to notice that in general equality will not hold. We have not succeeded to construct an example with $S(\mathscr{A} \otimes \mathscr{B}) < H(\mathscr{A}) + H(\mathscr{B} \mid \mathscr{A})$ but we have found an example with $S(\mathscr{A} \otimes \mathscr{B}) > H(\mathscr{A}) + H(\mathscr{B} \mid \mathscr{A})$. In this example we can control $H(\mathscr{B} \mid \mathscr{A})$. Classically this is always $\geqslant 0$ and $= 0$ iff $\omega$ restricted to $\mathscr{B}$ is pure. In the quantum situation it is $= 0$ again if restricted to $\mathscr{B}$ it is pure, but also if restricted to $\mathscr{A} \otimes \mathscr{B}$ it is pure. But as we will see this does not cover all possibilities, it can be $= 0$ though $\mathscr{B}$ seems to contain additional information.

**Example.** Let $\mathscr{A} = M_2$, $\mathscr{B} = M_2$ with matrix units $e_{ij} \in \mathscr{A}$ and matrix units $f_{ij} \in \mathscr{B}$. Let $\omega$ be given by the density matrix in $\mathscr{A} \otimes \mathscr{B}$

$$\rho = a/2\, |e_1 \otimes f_1 + e_2 \otimes f_2\rangle\langle e_1 \otimes f_1 + e_2 \otimes f_2| \\ + (1-a)\, |e_1 \otimes f_2\rangle\langle e_1 \otimes f_2| = a\, |\psi_1\rangle\langle\psi_1| + (1-a)\, |\psi_2\rangle\langle\psi_2|.$$

Reduced to $\mathscr{A}$ resp. $\mathscr{B}$ we have $\omega(e_{11}) = \omega(f_{22}) = 1 - a/2, \omega(e_{12}) = \omega(f_{12}) = 0, \omega(e_{22}) = \omega(f_{11}) = a/2$. Any state contributing to a decomposition of $\omega$ has the form

$$\sigma = \alpha\, |\psi_1\rangle\langle\psi_1| + \beta\, |\psi_2\rangle\langle\psi_2| + \gamma\, |\psi_1\rangle\langle\psi_2| + \gamma^*\, |\psi_2\rangle\langle\psi_1|.$$

Calculating its reduction to $\mathscr{A}$ resp. $\mathscr{B}$ we get again $\omega(e_{11}) = \omega(f_{22}), \omega(e_{12}) = \omega(f_{21})$, so that $S(\sigma, \mathscr{A}) = S(\sigma, \mathscr{B})$. Therefore $H(\mathscr{A} \mid \mathscr{B}) = H(\mathscr{B} \mid \mathscr{A}) = 0$.

## REFERENCES

1. H. Araki and E. H. Lieb, *Comm. Math. Phys.* **18**:160–170 (1970).
2. E. H. Lieb, *Bull. Amer. Math. Soc.* **81**:1–13 (1975).
3. E. H. Lieb and M. B. Ruskai, *J. Math. Phys.* **14**:1938–1941 (1973).
4. E. H. Lieb and M. B. Ruskai, *Phys. Rev. Lett.* **30**:434–436 (1973).
5. M. Ohia and D. Petz, *Quantum Entropy and its Use* (Springer, Berlin, 1993).
6. A. S. Holevo, *Problems Inform. Transmission* **15**:247–253 (1979).
7. A. Connes, H. Narnhofer, and W. Thirring, *Comm. Math. Phys.* **112**:691–719 (1987).
8. C. H. Bennett, D. V. Di Vincenzo, J. Smolin, and W. K. Wootters, *Phys. Rev. A* **54**:3824 (1996).
9. R. A. Bertlmann, H. Narnhofer, and W. Thirring, *Phys. Rev. A* **66**:033219 (2002).
10. M. Horodecki, P. Horodecki, and R. Horodecki, in *Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments*, Tracts in Modern Physics, Vol. 173 (Springer, 2001), pp. 151–191, quant-ph/0109124.
11. V. Vedral and M. B. Plenio, *Phys. Rev. Lett.* **78**:2275–2279 (1997); *Phys. Rev. A* **57**:1619–1633 (1998).
12. P. Hayden, R. Jozsa, D. Petz, and A. Winter, *Structure of States which Satisfy Strong Subadditivity of Quantum Entropy with Equality,* quant-ph/0304007.